

DTP/95/61
 hep-th/9510202
 October 1995

COSMIC p -BRANES

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ABSTRACT

We consider the metrics for cosmic strings and p -branes in spacetime dimension $N > 4$, that is, we look for solutions to Einstein-Maxwell-Dilaton gravity in N -dimensions with boost symmetry in the p -directions along the brane. Focussing first in detail on the five dimensional uncharged cosmic string we discuss the solution, which turns out to have a naked singularity on the brane, as well as considering its Kaluza-Klein reduction. We show how singularities may be avoided with particular core models. We then derive the general uncharged p -brane solution in arbitrary dimension. Finally, we consider an Einstein-Maxwell-Dilaton action, with arbitrary value of the dilaton coupling parameter, deriving the solutions for electrically and magnetically charged branes, as well as a class of self-dual branes.

PACS number: 97.60.Lf, 04.20.Jb, 11.10.Lm

Keywords: p -branes, Kaluza-Klein, gravity, string theory, topological defects

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1. Introduction.

Although it is true that on the large scale we seem to live in a four-dimensional world, the idea that spacetime might have more than four dimensions, with those extra dimensions curled up on a very small scale, has always had some attraction, most recently in the context of string theory. From a gravitational point of view, one of the interesting features of having more dimensions is that event horizons can have more interesting topologies. Instead of being restricted to spherical, or hyperspherical, surfaces, exact solutions exist [1,2] which have extended event horizons such as $S^2 \times \mathbb{R}^7$ etc. In fact, any vacuum solution to Einstein's equations in N -dimensions can be trivially extended to an $(N+1)$ -dimensional solution simply by adding in a flat direction. The solutions of Gibbons, Maeda, Horowitz and Strominger (GMHS) are less trivial in that they represent charged extended objects in Einstein-dilaton gravity, however one notable feature of these solutions is their lack of boost symmetry. The uncharged 'brane' has a composite metric of the aforementioned form: a D -dimensional black hole times a p -dimensional flat euclidean space. Although the metric of the charged brane is more complex, the magnetically charged branes with a specific dilaton coupling appropriate to low energy string gravity do take such a form[2]. Curiously, for all brane types, boost symmetry is restored in the extremal limit. Another interesting feature of the GMHS solutions is that most of the exact solutions are unstable [3], excepting those that are extremal [4]. Since the energy-momentum of the charge-field and dilaton does have boost symmetry, it is tempting to suggest that these instabilities are connected with the fact that the spacetime symmetries do not correspond to the source symmetries. This leaves us with the question: How does the metric of a boost symmetric brane behave?

From a completely different viewpoint, in cosmology we are often interested in the gravitational properties of cosmic strings and other defects, since these may have relevance for structure formation in the early universe (for recent reviews see [5] and [6]). If there are extra dimensions, it is interesting to query how these might effect the metric of a cosmic string, say. For example, suppose spacetime is topologically $\mathbb{R}^4 \times S^1$, as in Kaluza-Klein theory. If our string is infinite in the \mathbb{R}^4 part of the spacetime, we might hope that the metric is essentially unaffected, and indeed one can see that this is the case from considering the four-dimensional Kaluza-Klein theory. However, what if the string winds around the internal S^1 ? This would appear to be a point-like, presumably uncharged, source and hence a Schwarzschild metric such as one might obtain from dimensionally reducing the solutions in [1,2]. However, as we will show, such an argument ignores the crucial feature of a cosmic string (and more general p -brane defects formed as solitons in some field theory) which is

the boost symmetry on the brane. The five-dimensional (uncompactified) Schwarzschild $\times \mathbb{R}$ metric cannot be the metric of a vortex in five dimensions precisely because it does not have boost symmetry. What then does a five-dimensional string look like?

Finally, it has recently been argued that cosmic strings might be unstable to black hole pair creation along their length[7-10]. Although the topology of the vacuum manifold of a cosmic string (or other defect) generally protects against decay; if one allows a *space-time* topology changing process, such as black hole pair creation, then the usual stability arguments can be sidestepped. Such a process appears to be completely democratic, if it can occur (see [10] for a discussion of caveats) then it will occur, whether or not the defect is of Bogomolnyi type*. The instability in four-dimensions relies on the C-metric[12], an exact solution of the vacuum Einstein equations representing two black holes uniformly accelerating apart, connected to infinity by a pair of conical deficit singularities. The conical singularity of the C-metric is then replaced by the snub-nosed cone of the cosmic string[10]. Suppose a similar decay process exists for higher dimensional strings and branes, then by the principle of democracy, both Bogomolnyi and non-Bogomolnyi defects would be susceptible to decay. However, a necessary first step to any such investigation is an understanding of the higher dimensional analogue of the conical singularity in four dimensions.

In this paper, we attempt to answer these questions. We look for metrics which have the form

$$(D\text{-Hole}) \times p\text{-boost directions}$$

and might therefore correspond to p -branes in general dimensions. We consider uncharged and charged branes in N -dimensional gravity. We find the general form of the metric which turns out generically to have a null naked singularity at the core. We will argue that this is a property only of the exact solution, and can be smoothed out by a core model, rather like the four-dimensional cosmic string core smooths out the apex of the conical singularity[13,14].

The layout of the paper is as follows. In the next section we focus on the five-dimensional uncharged string. We derive the metric, discussing the singularity and core models, finally considering the Kaluza-Klein reduction of our solution. In section three we generalise our results to uncharged p -branes in N -dimensions. In section four, we

* We are using the term Bogomolnyi defect in the usual sense here, in that it refers to a specific field theory in which the second order equations can be reduced to first order by virtue of a special choice of coupling constants[11]. Such field theories are generally supersymmetrizable.

consider the charged five-dimensional string, using an Einstein-Maxwell-Dilaton gravity with arbitrary dilaton coupling. We consider the general case in section five, giving a few string-motivated examples. We then consider the electrically charged solutions and a special class of self-dual solutions. Finally we sum up our results.

2. The Five-Dimensional String.

Let us start by examining a string in five dimensions since this is the first non-trivial scenario to consider, and it is useful for visualisation. We first look for a vacuum solution, since any real cosmic string ought to asymptote this form.

One can assume (or show, with reference to a specific core model) that the 5-dimensional boost symmetric metric will take the form

$$ds^2 = A^2(dt^2 - dz^2) - B^2dr^2 - C^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.1)$$

which has the vacuum Einstein equations

$$R_0^0 = B^{-2} \left[\frac{A''}{A} - \frac{A'B'}{AB} + \left(\frac{A'}{A} \right)^2 + 2 \frac{A'C'}{AC} \right] = 0 \quad (2.2a)$$

$$R_r^r = 2B^{-2} \left[\frac{A''}{A} + \frac{C''}{C} - \frac{B'}{B} \left(\frac{A'}{A} + \frac{C'}{C} \right) \right] = 0 \quad (2.2b)$$

$$R_\theta^\theta = B^{-2} \left[\frac{C''}{C} + \frac{C'}{C} \left(\frac{C'}{C} + 2 \frac{A'}{A} - \frac{B'}{B} \right) \right] - \frac{1}{C^2} = 0. \quad (2.2c)$$

Obviously there is still coordinate freedom in the metric (2.1), and we choose to restrict it in part by setting

$$B = A^{-2}. \quad (2.3)$$

This somewhat unusual choice was motivated by trying to take account of the extra “ dz^2 ” piece multiplying A^2 , but turns out to give the simplest form for the solution. Obviously we need boundary conditions in order to solve (2.2), and since we are finding an asymptotic solution we impose boundary conditions at infinity, demanding that the spacetime be asymptotically Minkowskian in the four-dimensional sections transverse to the string, in other words

$$C \sim r \quad ; \quad A \rightarrow 1 \quad \text{as } r \rightarrow \infty. \quad (2.4)$$

With the substitution (2.3), the Einstein equations can be written in the form

$$((A^4)'C^2)' = 0 \quad (2.5a)$$

$$\left(\frac{A'}{A} \right)^2 + \left(\frac{C'}{C} \right)^2 + \left(\frac{A'C'}{AC} \right) = \frac{1}{C^2} \quad (2.5b)$$

$$(A^4(C^2)')' = 2. \quad (2.5c)$$

Hence by direct integration

$$(A^4)'C^2 = 4a_0 \quad (2.6a)$$

$$A^4(C^2)' = 2r + 2c_0 \quad (2.6b)$$

and thence

$$A^4C^2 = r^2 + 2(c_0 + 2a_0)r + b_0 = (r - r_+)(r - r_-), \quad (2.6c)$$

where (2.5b) gives

$$b_0 = c_0^2 + a_0^2 + 4a_0c_0 \Rightarrow r_{\pm} = -c_0 - 2a_0 \pm \sqrt{3}a_0. \quad (2.7)$$

Choosing the origin of our r -coordinate to set $r_- = 0$ ($c_0 = -(2 + \sqrt{3})a_0$), it is not difficult to show that the metric of the string is given by

$$ds^2 = \left(1 - \frac{r_+}{r}\right)^{\frac{1}{\sqrt{3}}} (dt^2 - dz^2) - \left(1 - \frac{r_+}{r}\right)^{\frac{-2}{\sqrt{3}}} dr^2 - r^2 \left(1 - \frac{r_+}{r}\right)^{1 - \frac{2}{\sqrt{3}}} (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.8)$$

What are the important features of this metric? First of all, it is asymptotically flat in the 4-dimensional sense, asymptoting

$$ds^2 \simeq \left(1 - \frac{r_+}{\sqrt{3}r}\right) (dt^2 - dz^2) - \left(1 + \frac{2r_+}{\sqrt{3}r}\right) dr^2 - r^2 \left(1 + \frac{(2 - \sqrt{3})r_+}{\sqrt{3}r}\right) d\Omega_H^2 \quad (2.9)$$

We find the ADM mass per unit for this metric is $r_+/\sqrt{3} = 2a_0$ whereas the gravitational mass is $r_+/2\sqrt{3} = a_0$. The metric is however singular as $r \rightarrow r_+$. This singularity is of a particularly unpleasant nature since it is both naked and has a divergent volume element. To see that the singularity is naked, consider an outgoing radial null geodesic for which

$$\frac{dr}{dt} = \left(1 - \frac{r_+}{r}\right)^{\sqrt{3}/2} \quad (2.10)$$

Therefore, for a null geodesic starting at r_+ at time t_+ , the geodesic reaches $r_+ + \delta r$ at time $t_+ + \delta t$ given by

$$\delta t \propto \delta r^{1 - \sqrt{3}/2} \quad (2.11)$$

which is certainly finite. However, since

$$g_{tt} \frac{dt}{d\lambda} = \left(1 - \frac{r_+}{r}\right)^{\frac{1}{\sqrt{3}}} \frac{dt}{d\lambda} = \text{constant} \quad (2.12)$$

for λ an affine parameter along the geodesic, any escaping photons are infinitely redshifted.

This property in itself might appear to rule out these solutions in situations of physical interest, however, an idealised (Nambu) string in four dimensions has a “naked singularity” – the conical singularity – that is not even null! It is only the gentle (integrable) nature of this singularity that makes us tolerate it, as well as the fact that it has fairly convincingly been shown to be a good approximation to the real thing[13,14]. Since we are only looking for an exterior solution to a five-dimensional cosmic string, we would hope that the core would somehow smooth out this unpleasant behaviour, rather like the core of a four-dimensional cosmic string smooths out the singular apex of the conical spacetime even in quite general scenarios. Reversing the logic, one might hope that this singular behaviour is simply the appropriate higher dimensional analogue of the conical singularity.

To provide evidence for this claim, we will consider a fairly general core model, whereby

$$T_b^a = \frac{1}{4\pi} \text{diag} \{E(r), E(r), -P_r(r), -P_\theta(r), -P_\theta(r)\}. \quad (2.13)$$

The dominant energy condition will be assumed ($E \geq |P_*|$), and the functions will be assumed to be effectively zero outside some finite core. This assumption will replace more specific fall-off conditions, such as those derived for the four-dimensional string[14], which would require a far more detailed analysis than is appropriate here - we merely wish to show that it is plausible to smooth out the singularity. We additionally make a weak field approximation, namely that

$$\mu = \int_0^\infty \sqrt{-g} E(r) dr \ll 1. \quad (2.14)$$

This approximation will mean that we can expand the equations of motion around flat space, and hopefully derive a consistent solution.

The flat space matter equation, $T^{ab}_{,b} = 0$, implies

$$(r^2 P_r)' = 2r P_\theta. \quad (2.15)$$

Since we are in five dimensions the Einstein equations read

$$R_b^a = 8\pi(T_b^a - \frac{1}{3}T\delta_b^a) \quad (2.16)$$

and hence (2.2a-c) can be rearranged to give

$$((A^4)'C^2)' = \frac{8C^2}{3}(E + P_r + 2P_\theta) \quad (2.17a)$$

$$\left(\frac{A'}{A}\right)^2 + \left(\frac{C'}{C}\right)^2 + 4\frac{A'C'}{AC} = \frac{1 + 2C^2 P_r}{A^4 C^2} \quad (2.17b)$$

$$(A^4 C^2)'' = 2 + 4C^2(P_r + P_\theta) \quad (2.17c)$$

assuming $B = A^{-2}$ as before. We will integrate these equations out from the core, making no assumption as to the asymptotic solution, although of course we wish to show that the asymptotic form of the metric is the vacuum solution (2.8), up to coordinate redefinition. We therefore impose the boundary conditions at the core

$$A = 1 \quad ; \quad C \sim r \quad \text{as } r \rightarrow 0. \quad (2.18)$$

Note that, even if we had not assumed *a priori* that $g_{zz} = g_{tt}$, the form of the energy-momentum tensor together with the above boundary conditions (with $g_{zz} = 1$ at $r = 0$) would have necessitated $g_{zz} = g_{tt}$.

From (2.17a,c) we can see that the integration constants a_0 , c_0 in (2.6) are given by

$$a_0 = \frac{2}{3}\mu \quad ; \quad c_0 = p - \frac{4}{3}\mu \quad (2.19)$$

where $p = \int_0^\infty r^2 P_r$. To lowest order, (2.17c) can actually be integrated up fully to give

$$A^4 C^2 = r^2 + 2r \int_0^r r^2 P_r. \quad (2.20)$$

The integrability constraint (2.17b) is then automatically satisfied to first order, and the first order solutions are given by

$$A(r) \simeq 1 - \frac{2}{3} \frac{\int_0^r r^2 E}{r} + \frac{2}{3} \int_0^r r(E + P_r) \quad (2.21a)$$

$$C(r) \simeq r \left(1 - \frac{4}{3} \int_0^r r(E + P_r) \right) + \int_0^r r^2 P_r + \int_0^r r^2 E \quad (2.21b)$$

We can then see, again to first order in μ , that the solution asymptotes (2.8) up to a rescaling.

Obviously this does not prove that we can smooth out the singularity, but it does at least provide encouragement that it might be possible to do so by judicious choice of source. The main problem in finding a topological defect source for the uncharged string is in the restriction to no long range interactions. Since the plane orthogonal to the vortex is now three-dimensional we are looking, in the context of defects from spontaneous symmetry breaking, for a vacuum with non-trivial second homotopy group. If we spontaneously break such a symmetry, we will not in general break it completely, for there is still a residual $U(1)$ symmetry around any point in the vacuum manifold. This translates to a residual long range interaction and hence probable charge for the defect. On the other hand, defects

formed from symmetry breaking are not the only types of soliton one could consider. It is possible that the topology of the field space itself might be suitable for the formation of an extended field configuration with finite energy per unit length. For example, the Skyrme model [15], which has localised finite energy field configurations in four dimensions, might be a good candidate for the uncharged string. The flat space solution would satisfy our energy-momentum conditions, and appears to be a promising source. However, one would have to check that the extended solution in five dimensions did not exhibit any obvious instabilities, as well as coupling the model to gravity - an involved problem even in four dimensions[16]. Nonetheless, with suitable disclaimers*, we believe that the arguments given are indicative that the singularity is an artifact of the vacuum solution and that real strings will have non-singular cores.

Since we are in five-dimensions, let us discuss the Kaluza-Klein reduction of our solution before proceeding to generalize it. In a conventional Kaluza-Klein reduction, the five-dimensional metric is written as

$$ds^2 = -e^{4\sigma/\sqrt{3}}(dz + 2A_a dx^a)^2 + e^{-2\sigma/\sqrt{3}} g_{ab} dx^a dx^b \quad (2.22)$$

which (after integration along z) yields an action

$$\int d^4x \sqrt{-g} \left[\frac{-R}{16\pi} - \frac{1}{4} e^{2\sqrt{3}\sigma} F_{ab}^2 + \frac{1}{8\pi} (\nabla\sigma)^2 \right] \quad (2.23)$$

as the effective four-dimensional action. Since we have our five-dimensional vacuum metric (2.8), we may read off:

$$e^{4\sigma} = \left(1 - \frac{r_+}{r}\right) \quad (2.24)$$

and

$$ds_4^2 = \left(1 - \frac{r_+}{r}\right)^{\sqrt{3}/2} dt^2 - \left(1 - \frac{r_+}{r}\right)^{-\sqrt{3}/2} dr^2 - r^2 \left(1 - \frac{r_+}{r}\right)^{1-\sqrt{3}/2} d\Omega_{II}^2 \quad (2.25)$$

this is again singular at $r = r_+$, although the volume element is slightly better behaved.

Such a four dimensional metric is not new, it is similar to the metrics of Brans and Dicke[17] (see also Dicke[18]) which were derived in their original paper on Mach's principle and gravitation, although Brans and Dicke considered the metrics in a conformally related frame, equivalent to simply truncating the five-dimensional metric (2.8). Additionally,

* e.g. <http://xxx.lanl.gov/legal/disclaimer.html>

Kaluza Klein theory corresponds to Brans-Dicke theory for $\omega = 0$, and therefore is technically outside the regime of the original Brans-Dicke results, however, provided one does not use Brans and Dicke's values of their solution parameters in terms of ω , which were derived in the far field $\omega > 3/2$ limit, the metrics can be seen to agree. Indeed these solutions have been more recently considered in [19] in a broader context, where it was argued that the naked singularities corresponded to a non-trivial Parametrized Post Newtonian γ parameter in the "Brans-Dicke" frame. The PPN parameters are used as a means of measuring how far a particular solution diverges from the Einstein far field theory (see [20] for a review). Experimentally $\gamma = 1 \pm .002$ for the Sun[21]. ($\gamma = 2$ for our truncated metric.) These solutions were rejected in [19] as being physically unacceptable; here however, we have obtained this metric by reduction of a five-dimensional object. This metric is the metric of a bosonic Nambu string winding mode in five dimensional Kaluza Klein theory and should certainly not be ignored within the rationale of string theory. Moreover, since we have argued that the unpleasant singularity can most probably be smoothed out by a core, this indicates that such sources should not be regarded as physically unacceptable from the four-dimensional point of view, but rather, should be regarded as being physically acceptable as dimensionally reduced solitons. Additionally, since we could hardly claim that the Sun is a Nambu string winding mode, the Viking data limits[21] on γ do not really rule out such solutions. Now let us consider more general brane metrics.

3. General p-Branes.

Following (2.1), the general boost symmetric metric of a p -brane in N -dimensions should have the form

$$ds^2 = A^2(dt^2 - dx_i dx^i) - B^2 dr^2 - C^2 d\Omega_{D-2}^2 \quad (3.1)$$

where $D = N - p$, and $i = 1, \dots, p$ runs over the brane coordinates. The vacuum Einstein equations are

$$R_0^0 = B^{-2} \left[\frac{A''}{A} - \frac{A'B'}{AB} + p \left(\frac{A'}{A} \right)^2 + (D-2) \frac{A'C'}{AC} \right] = 0 \quad (3.2a)$$

$$R_r^r = B^{-2} \left[(p+1) \frac{A''}{A} + (D-2) \frac{C''}{C} - \frac{B'}{B} \left((p+1) \frac{A'}{A} + (D-2) \frac{C'}{C} \right) \right] = 0 \quad (3.2b)$$

$$R_\theta^\theta = B^{-2} \left[\frac{C''}{C} + \frac{C'}{C} \left((D-3) \frac{C'}{C} + (1+p) \frac{A'}{A} - \frac{B'}{B} \right) \right] - \frac{(D-3)}{C^2} = 0. \quad (3.2c)$$

Since $B = A^{-2}$ was helpful in solving the five-dimensional cosmic string, we will set $B = A^{-n}$ and search for a solution in a similar fashion here.

As before, we can directly integrate (3.2a) to obtain

$$(A^{p+n+1})'C^{D-2} = a_0(p+n+1). \quad (3.3)$$

However, (3.2c) is no longer directly integrable. Instead, based on the intuition gleaned from the five dimensional string, we try

$$A^{p+n+1} = \left(1 - \left(\frac{r_+}{r}\right)^{D-3}\right)^m \quad (3.4)$$

which gives immediately that

$$C^{D-2} = r^{D-2} \left(1 - \left(\frac{r_+}{r}\right)^{D-3}\right)^{1-m} \quad (3.5)$$

from (3.3) and hence we see that our spacetime is asymptotically flat in the D-dimensions orthogonal to the brane. Substituting these forms into (3.2b,c) give two relations on m and n which, after some algebra, can be solved to give

$$n = \frac{p+1}{D-3} + \frac{D-4}{D-3} \sqrt{\frac{(p+1)(D+p-2)}{(D-2)}} \quad (3.6a)$$

$$m = \frac{D-4}{2(D-3)} + \frac{D-2}{2(D-3)} \sqrt{\frac{(p+1)(D-2)}{D+p-2}}. \quad (3.6b)$$

Thus the general solution is given by (3.4), (3.5) and (3.6) with $B = A^{-n}$. By setting $D = 4$, $p = 1$, we obtain the string of the previous section. To illustrate the solution, we will consider two examples, the 5-brane in ten dimensions and the p -branes in $4+p$ dimensions.

The 5-brane.

The 5-brane in 10 dimensions has $p = D = 5$, therefore (3.6) gives $n = 5$, $m = 11/8$, and hence a metric:

$$ds_5^2 = \left(1 - \frac{r_+^2}{r^2}\right)^{\frac{1}{4}} (dt^2 - dx_i^2) - \left(1 - \frac{r_+^2}{r^2}\right)^{-\frac{5}{4}} dr^2 - r^2 \left(1 - \frac{r_+^2}{r^2}\right)^{-\frac{1}{4}} d\Omega_{III}^2 \quad (3.7)$$

This again has a naked singularity at $r = r_+$, but should be smoothable by a procedure analogous to that described in the previous section.

The p -branes for $D = 4$.

Another simpler family of solutions are the p -branes in $4 + p$ dimensions for which $n = 1 + p$ and $m = \sqrt{2(p+1)/(2+p)}$. Here

$$ds^2 = \left(1 - \frac{r_+}{r}\right)^{\frac{\sqrt{2}}{\sqrt{(p+1)(p+2)}}} (dt^2 - dx_i^2) - \left(1 - \frac{r_+}{r}\right)^{\frac{\sqrt{2(p+1)}}{\sqrt{(2+p)}}} dr^2 - r^2 \left(1 - \frac{r_+}{r}\right)^{1 - \sqrt{\frac{2(p+1)}{(2+p)}}} d\Omega_{II}^2 \quad (3.8)$$

We may perform a Kaluza-Klein reduction by setting

$$ds_N^2 = -\sigma^2 dx_i^2 + \sigma^{-p} ds_4^2 \quad (3.9)$$

which yields the four-dimensional metric

$${}^4ds^2 = \left(1 - \frac{r_+}{r}\right)^{\sqrt{\frac{p+2}{2(p+1)}}} dt^2 - \left(1 - \frac{r_+}{r}\right)^{-\sqrt{\frac{p+2}{2(p+1)}}} dr^2 - r^2 \left(1 - \frac{r_+}{r}\right)^{1 - \sqrt{\frac{p+2}{2(p+1)}}} d\Omega_{II}^2 \quad (3.10)$$

which gives a slightly different four-dimensional behaviour for each p , and a PPN parameter $\gamma = (p+1)$ in the ‘‘Brans-Dicke’’ frame.

4. The Charged String.

We now wish to generalize the work of the previous two sections to include the effect of charge. We will begin by setting up the general formalism before specializing to the five-dimensional string in this section. The next section will deal with the general branes. We consider an action similar to that of Horowitz and Strominger[2], (see also [22] and references therein for a review of string solitons) namely an ‘‘Einstein-Maxwell-Dilaton’’ action with arbitrary dilaton coupling in N -dimensions:

$$S = \int d^N x \sqrt{-\tilde{g}} \left\{ e^{-2\phi} [-\tilde{R} - 4(\tilde{\nabla}\phi)^2] (-)^{D-3} \frac{2e^{2a\phi} F^2}{(D-2)!} \right\} \quad (4.1)$$

Here, F is a $(D-2)$ -form, ϕ the dilaton and a the dilaton coupling. However, rather than considering solving the field equations in the string frame, we will make a conformal transformation to the Einstein frame where our analysis will follow more closely the previous two sections.

We therefore define

$$g_{ab} = e^{\frac{-4\phi}{(N-2)}} \tilde{g}_{ab} \quad (4.2)$$

which gives us an action in Einstein form

$$S = \int d^N x \sqrt{-g} \left\{ -R + \frac{4}{N-2} (\nabla\phi)^2 (-)^{D-3} \frac{2F^2 e^{2\alpha\phi}}{(D-2)!} \right\} \quad (4.3)$$

where

$$\alpha = a + \frac{p+4-D}{N-2} \quad (4.4)$$

gives the shifted dilaton coupling in the Einstein frame. In this format, we can use the previous Einstein equations with the source

$$T_{ab} = \frac{4}{N-2} \nabla_a \phi \nabla_b \phi (-)^{D-3} \frac{2e^{2\alpha\phi}}{(D-3)!} F_{a\dots} F_b \dots - g_{ab} \left[\frac{2}{N-2} (\nabla\phi)^2 (-)^{D-3} \frac{F^2 e^{2\alpha\phi}}{(D-2)!} \right] \quad (4.5)$$

We also have the dilaton and electromagnetic equations of motion:

$$\nabla_a [e^{2\alpha\phi} F^{a\dots}] = 0 \quad (4.6a)$$

$$\square\phi = (-)^{D-3} \frac{\alpha(N-2)}{2(D-2)!} e^{2\alpha\phi} F^2 \quad (4.6b)$$

Following Horowitz and Strominger, we will first look for a magnetically charged solution

$$F = Q \epsilon_{D-2} \quad (4.7)$$

where ϵ_{D-2} is the area form of a unit (D-2)-sphere. We will also assume that $\phi = \phi(r)$. This form of F then automatically solves (4.6a). With these assumptions, and using the boost symmetric form of the metric (3.1), the energy momentum tensor takes the form

$$T_0^0 = \frac{2}{N-2} \frac{\phi'^2}{B^2} + \frac{Q^2 e^{2\alpha\phi}}{C^{2(D-2)}} \quad (4.8a)$$

$$T_r^r = -\frac{2}{N-2} \frac{\phi'^2}{B^2} + \frac{Q^2 e^{2\alpha\phi}}{C^{2(D-2)}} \quad (4.8b)$$

$$T_\theta^\theta = \frac{2}{N-2} \frac{\phi'^2}{B^2} - \frac{Q^2 e^{2\alpha\phi}}{C^{2(D-2)}} \quad (4.8c)$$

and the dilaton equation

$$\left(\frac{A^{p+1} C^{D-2} \phi'}{B} \right)' = \frac{(N-2)\alpha}{2} \frac{Q^2 e^{2\alpha\phi} A^{p+1} B}{C^{D-2}} \quad (4.9)$$

Substituting $T_b^a - \frac{T}{N-2}\delta_b^a$ as the source in the RHS of (3.2) gives the Einstein equations

$$\frac{(A^p A' C^{D-2}/B)'}{A^{p+1} C^{D-2} B} = 2 \frac{(D-3)}{(N-2)} \frac{Q^2 e^{2\alpha\phi}}{C^{2(D-2)}} \quad (4.10a)$$

$$\frac{(A^{p+1} C^{D-3} C'/B)'}{A^{p+1} C^{D-2} B} - \frac{D-3}{C^2} = -2 \frac{(p+1)}{(N-2)} \frac{Q^2 e^{2\alpha\phi}}{C^{2(D-2)}} \quad (4.10b)$$

$$\frac{(1+p)}{AB} \left(\frac{A'}{B}\right)' + \frac{(D-2)}{CB} \left(\frac{C'}{B}\right)' = -4 \frac{\phi'^2}{B^2(N-2)} + 2 \frac{(D-3)}{(N-2)} \frac{Q^2 e^{2\alpha\phi}}{C^{2(D-2)}} \quad (4.10c)$$

Finally, we impose similar boundary conditions as in the uncharged branes, namely that spacetime be asymptotically Minkowskian, $A, B \rightarrow 1$, $C \sim r$ at infinity. Additionally, for ϕ we will impose that $\phi \rightarrow 0$ as $r \rightarrow \infty$ and that ϕ' is regular at the event horizon except possibly in some extremal limit. The former boundary condition on ϕ is merely a choice for algebraic simplicity, the latter boundary condition ensures that the weak field limit $Q \ll M$ corresponds to a perturbation of the charge free brane outside the event horizon.

These are obviously considerably more complicated than the uncharged branes, so once more we will solve for the five-dimensional string as a warm-up, before trying to tackle the full problem. Therefore, we set $N = 5$, $D = 4$ and $p = 1$, and as before, look for a solution with set $B = A^{-2}$. The reason for choosing the Einstein frame now becomes slightly more apparent: in four dimensions, for black holes in the string frame the time and radial parts of the metric react differently to charge[23], but react in the same way in the Einstein frame. Therefore, in five-dimensions, we might hope that in the Einstein frame we will have similar behaviour to the uncharged branes and therefore that we can still use some of the results from the previous two sections.

With these substitutions, and some minor shuffling, the system of equations we must solve is given by

$$((A^4)'C^2)' = \frac{16}{9\alpha}(A^4 C^2 \phi')' \quad (4.11a)$$

$$(A^4(C^2)')' = 2 - \frac{16}{9\alpha}(A^4 C^2 \phi')' \quad (4.11b)$$

$$\left[\frac{(A^4 C^2)'}{2A^4 C^2}\right]^2 - 3\left(\frac{A'}{A}\right)^2 = \frac{1}{A^4 C^2} + \frac{2}{3}\phi'^2 - \frac{2}{3\alpha} \frac{(A^4 C^2 \phi')'}{A^4 C^2} \quad (4.11c)$$

$$(A^4 C^2 \phi')' = \frac{3\alpha}{2} \frac{Q^2 e^{2\alpha\phi}}{C^2} \quad (4.11d)$$

subject to the aforementioned boundary conditions.

We may proceed analogously to section two, (2.6a-c), to obtain

$$(A^4)'C^2 = \frac{16}{9\alpha}A^4C^2\phi' + 4a_0 \quad (4.12a)$$

$$A^4(C^2)' = 2r + 2c_0 - \frac{16}{9\alpha}A^4C^2\phi' \quad (4.12b)$$

$$A^4C^2 = r^2 + 2(c_0 + 2a_0)r + b_0 = (r - r_+)(r - r_-). \quad (4.12c)$$

However, the integrability condition (4.11c) does not give directly a relation for b_0 . Instead it gives an equation for ϕ , which, writing $f = A^4C^2\phi'$, can be seen to be a Ricatti equation for f :

$$\begin{aligned} f' &= \frac{(\alpha + \frac{8}{9\alpha})f^2 + 4a_0f - \frac{3\alpha}{2}(c_0^2 + a_0^2 + 4a_0c_0 - b_0)}{A^4C^2} \\ &= \frac{\beta f^2 + 4a_0f + \gamma}{(r - r_+)(r - r_-)} \end{aligned} \quad (4.13)$$

Now, regularity of ϕ' implies $\gamma = 0$, and hence we have the same roots r_{\pm} as for the uncharged string. (4.13) is then readily integrated to give

$$f = \frac{4a_0f_{\infty}(r - r_+)^{2/\sqrt{3}}}{(\beta f_{\infty} + 4a_0)(r - r_-)^{2/\sqrt{3}} - \beta f_{\infty}(r - r_+)^{2/\sqrt{3}}} \quad (4.14)$$

where

$$f_{\infty} = \lim_{r \rightarrow \infty} A^4C^2f' = \frac{1}{\beta} \left(-2a_0 + \sqrt{4a_0^2 + 3\alpha\beta Q^2/2} \right). \quad (4.15)$$

using the equation of motion for ϕ . We can then proceed with the integration of the equations of motion (4.12a,b) to obtain in turn

$$Ae^{-\frac{4\phi}{9\alpha}} = \left(\frac{r - r_+}{r - r_-} \right)^{\frac{1}{2\sqrt{3}}} \quad (4.16a)$$

$$Ce^{\frac{8\phi}{9\alpha}} = (r - r_+)^{1 - \frac{2}{\sqrt{3}}} (r - r_-)^{1 + \frac{2}{\sqrt{3}}} \quad (4.16b)$$

$$e^{-\beta\phi} = 1 + \frac{\beta f_{\infty}}{4a_0} - \frac{\beta f_{\infty}}{4a_0} \left(\frac{r - r_+}{r - r_-} \right)^{\frac{2}{\sqrt{3}}}. \quad (4.16c)$$

Finally, we use the remaining coordinate freedom (choice of r-origin) to set $r_- = \beta f_{\infty}$ which implies

$$Q^2 = \frac{r_+ r_-}{4\alpha\beta}, \quad (4.17)$$

and gives the five dimensional boost-symmetric dilatonic charged string metric

$$\begin{aligned}
g_{tt} &= \left[1 + \frac{\sqrt{3}r_-}{2(r_+ - r_-)} - \frac{\sqrt{3}r_-}{2(r_+ - r_-)} \left(\frac{r - r_+}{r - r_-} \right)^{\frac{2}{\sqrt{3}}} \right]^{\frac{-8}{9\alpha\beta}} \left(\frac{r - r_+}{r - r_-} \right)^{\frac{1}{\sqrt{3}}} \\
g_{\theta\theta} &= \left[1 + \frac{\sqrt{3}r_-}{2(r_+ - r_-)} - \frac{\sqrt{3}r_-}{2(r_+ - r_-)} \left(\frac{r - r_+}{r - r_-} \right)^{\frac{2}{\sqrt{3}}} \right]^{\frac{16}{9\alpha\beta}} (r - r_-)^{1+\frac{2}{\sqrt{3}}} (r - r_+)^{1-\frac{2}{\sqrt{3}}} \\
g_{zz} &= g_{tt} \quad ; \quad g_{rr} = (g_{tt})^{-2} \quad ; \quad g_{\phi\phi} = \sin^2 \theta g_{\theta\theta}
\end{aligned} \tag{4.18}$$

It is interesting to examine this metric for a couple of special cases:

i) $\alpha = 0$

For $\alpha = 0$ the dilaton completely decouples from electromagnetism, and what we are left with is Einstein-Maxwell gravity in five dimensions. Noting that $\alpha\beta = 8/9$, we see that the charged string has metric

$$ds^2 = A^2(dt^2 - dz^2) - A^{-4}[dr^2 + (r - r_+)(r - r_-)d\Omega_H^2] \tag{4.19}$$

where

$$A^2 = \left[1 + \frac{\sqrt{3}r_-}{2(r_+ - r_-)} \left(1 - \left(\frac{r - r_+}{r - r_-} \right)^{2/\sqrt{3}} \right) \right]^{-1} \left(\frac{r - r_+}{r - r_-} \right)^{1/\sqrt{3}}. \tag{4.20}$$

We may compare this with the metric obtained by ignoring boost symmetry, and merely extending the magnetically charged Reissner Nordstrom solution by adding a flat direction:

$$ds^2 = \left(1 - \frac{r_+}{r} \right) \left(1 - \frac{r_-}{r} \right) dt^2 - dz^2 - \left(1 - \frac{r_+}{r} \right)^{-1} \left(1 - \frac{r_-}{r} \right)^{-1} dr^2 - r^2 d\Omega_H^2 \tag{4.21}$$

Clearly these two metrics are rather different! However, if one were to consider the metric as arising from, say an $SU(2)$ monopole in five dimensions, then the boost symmetry of the energy-momentum tensor would require the metric (4.20) rather than (4.21). The only remaining question is whether the equation of motion would integrate out to this asymptotic form.

It is beyond the scope of this paper to analyse the non-linear field equations resulting from such a substitution, however, by referring to the first order corrections to the metric (2.21) obtained for a rather general energy-momentum tensor, we can see that provided the defect is weakly gravitating in the sense of (2.14), the $1/r^2$ fall-off typical of a charged source should not obstruct the integrals in (2.21) remaining finite, and the indications are

that the metric will indeed integrate out to the asymptotic form (4.25). We can make some observations on the BPS limit, in this limit, the potential of the Higgs field vanishes and the equations of motion become first order in the absence of gravity. The now massless scalar acquires a long range fall-off which exactly counterbalances the electromagnetic fall-off. The energy-momentum tensor in flat space, as expected, has no non-zero components orthogonal to the monopole worldsheet. Without reference to the specific fields, one can see that coupling in gravity destroys this balance, as it does in four-dimensions[24,25]. This can be understood as a consequence of the attractive nature of gravity requiring some radial pressure to support against collapse.

The extremal limit of (4.19) takes the form

$$\left(1 - \frac{r_+}{r}\right) (dt^2 - dz^2) - \left(1 - \frac{r_+}{r}\right)^{-2} dr^2 - r^2 d\Omega_H^2 \quad (4.22)$$

which does not appear to be singular as $r \rightarrow r_+$. However, for $r < r_+$ something curious occurs. Normally, it is the time and radial coordinates that swap roles across the event horizon, r becoming timelike, and t spacelike. However, in the above metric, it appears to be t and z that are trading places, i.e., the worldsheet coordinates.

ii) $\alpha = -2/3$

This special case corresponds to $a = -1$, or a dilaton coupling usually associated with low energy string gravity. For $\alpha = -2/3$, $\beta = -2$, and therefore we have

$$e^{2\phi} = \left[1 + \frac{\sqrt{3}r_-}{2(r_+ - r_-)} - \frac{\sqrt{3}r_-}{2(r_+ - r_-)} \left(\frac{r - r_+}{r - r_-} \right)^{\frac{2}{\sqrt{3}}} \right] \quad (4.23)$$

By noting that $\frac{8}{9\alpha\beta} = \frac{4}{3}$ we have the exponents for the metric in (4.18), however, it is more enlightening to transform back to the string frame, multiplying by a conformal factor of $e^{4\phi/3}$ to obtain the metric in the string frame:

$$\begin{aligned} d\tilde{s}^2 &= \left(\frac{r - r_+}{r - r_-} \right)^{\frac{1}{\sqrt{3}}} (dt^2 - dz^2) - e^{4\phi} \left(\frac{r - r_+}{r - r_-} \right)^{\frac{-4}{\sqrt{3}}} [dr^2 + (r - r_+)(r - r_-)d\Omega_H^2] \\ &\rightarrow dt^2 - dz^2 - \left(1 - \frac{r_+}{r} \right)^{-2} dr^2 - r^2 d\Omega_H^2 \quad \text{as } r_- \rightarrow r_+ \end{aligned} \quad (4.24)$$

This will be recognised as the truncated extremal limit of the Horowitz-Strominger 6-brane in 10 dimensions.

Now that we have an idea of the steps involved, we will tackle the general brane.

5. General Charged Branes.

We would now like to solve for the general charged p -brane. As before, we will begin by assuming that the brane is magnetically charged. In the next section we will show how to derive an electrically charged brane via a duality transformation.

Clearly (4.9) and (4.10a) imply

$$\frac{A^p A' C^{D-2}}{B} = a_0 + \frac{4(D-3)}{\alpha(N-2)^2} \phi' \frac{A^{p+1} C^{D-2}}{B} \quad (5.1)$$

and we might expect that, just as the five dimensional string metric maintains the same powers of $(1 - r_+/r)$, the general brane might be similarly related to the uncharged brane, and look for a solution

$$A^{p+n+1} = \left(\frac{r^{D-3} - r_+^{D-3}}{r^{D-3} - r_-^{D-3}} \right)^m \exp \left\{ \frac{4(D-3)(p+n+1)\phi}{\alpha(N-2)^2} \right\} \quad (5.2)$$

with m given by (3.6b). However, the substitution $B = A^{-n}$ rapidly leads to problems! Therefore, we relax the idea that the time and radial components of the metric react the same way to r_- , and instead try a substitution of the form

$$B = A^{-n} e^{\gamma\phi} \left(1 - \left(\frac{r_-}{r} \right)^{D-3} \right)^s \quad (5.3)$$

where n is given by (3.6a). Substituting these guesses into (5.1) gives

$$C^{D-2} = r^{D-2} \left(1 - \left(\frac{r_+}{r} \right)^{D-3} \right)^{1-m} \left(1 - \left(\frac{r_-}{r} \right)^{D-3} \right)^{1+m+s} \exp \left\{ \gamma\phi - \frac{4(p+n+1)(D-3)\phi}{\alpha(N-2)^2} \right\} \quad (5.4)$$

It is then straightforward, if lengthy, to show that these functions satisfy (4.10b) provided

$$s = -\frac{D-4}{D-3} \quad (5.5a)$$

$$\gamma = \frac{4}{\alpha(N-2)^2} [n(D-3) - (p+1)] \quad (5.5b)$$

Note that for $D = 4$, $\gamma = s = 0$ in agreement with the solution already derived for the five dimensional string.

Finally, writing

$$g = \frac{A^{p+1} C^{D-2}}{B} \quad ; \quad f = g\phi' \quad (5.6)$$

and substituting (5.2-4) into (4.10c) yields, as in the case of the five dimensional string, a Ricatti equation for f

$$gf' = \beta f^2 + l(D-3)(r_+^{D-3} - r_-^{D-3})f \quad (5.7)$$

where

$$\beta = \alpha + \frac{4(D-3)(p+1)}{\alpha(N-2)^2} \quad (5.8)$$

generalises the β of equation (4.13), and

$$l = \frac{2m(p+1)}{p+n+1} = \sqrt{\frac{(p+1)(D-2)}{(N-2)}}. \quad (5.9)$$

Additionally, (4.9) gives

$$gf' = \frac{(N-2)\alpha Q^2}{2} e^{2\beta\phi} \left(\frac{r_+^{D-3} - r_-^{D-3}}{r_+^{D-3} - r_-^{D-3}} \right)^l \quad (5.10)$$

One can then verify that

$$e^{-\beta\phi} = 1 + \frac{r_-^{D-3}}{l(r_+^{D-3} - r_-^{D-3})} \left[1 - \left(\frac{r_+^{D-3} - r_-^{D-3}}{r_+^{D-3} - r_-^{D-3}} \right)^l \right] \quad (5.11)$$

is the solution satisfying these two equations and the boundary conditions, with the charge, Q , being given by

$$Q^2 = \frac{2(D-3)^2 r_-^{D-3}}{\alpha\beta(N-2)} (l(r_+^{D-3} - r_-^{D-3}) + r_-^{D-3}). \quad (5.12)$$

To summarize, the metric of the charged p -brane is given by

$$\begin{aligned} ds^2 = & e^{\frac{8(D-3)\phi}{\alpha(N-2)^2}} \left(\frac{r_+^{D-3} - r_-^{D-3}}{r_+^{D-3} - r_-^{D-3}} \right)^{\sqrt{\frac{D-2}{(p+1)(N-2)}}} (dt^2 - dx_i^2) \\ & - e^{\frac{-8(p+1)\phi}{\alpha(N-2)^2}} \left(\frac{r_+^{D-3} - r_-^{D-3}}{r_+^{D-3} - r_-^{D-3}} \right)^{-\frac{1}{D-3} \left[\sqrt{\frac{(p+1)(D-2)}{(N-2)}} + (D-4) \right]} \left(1 - \left(\frac{r_-}{r} \right)^{D-3} \right)^{\frac{-2(D-4)}{D-3}} dr^2 \\ & - e^{\frac{-8(p+1)\phi}{\alpha(N-2)^2}} \left(\frac{r_+^{D-3} - r_-^{D-3}}{r_+^{D-3} - r_-^{D-3}} \right)^{\frac{1}{D-3} \left[1 - \sqrt{\frac{(p+1)(D-2)}{(N-2)}} \right]} \left(1 - \left(\frac{r_-}{r} \right)^{D-3} \right)^{\frac{2}{(D-3)}} r^2 d\Omega_{D-2}^2 \end{aligned} \quad (5.13)$$

with the dilaton given by (5.11). An alternative form of the metric which is also useful is given by making the coordinate change

$$\begin{aligned} y^{D-3} &= r^{D-3} - r_-^{D-3} \\ y_0^{D-3} &= r_+^{D-3} - r_-^{D-3} \end{aligned} \quad (5.14)$$

in which case the metric has the form

$$\begin{aligned} ds^2 &= e^{\frac{8(D-3)\phi}{\alpha(N-2)^2}} \left(1 - \left(\frac{y_0}{y} \right)^{D-3} \right)^{\sqrt{\frac{D-2}{(p+1)(N-2)}}} (dt^2 - dx_i^2) \\ &\quad - e^{\frac{-8(p+1)\phi}{\alpha(N-2)^2}} \left(1 - \left(\frac{y_0}{y} \right)^{D-3} \right)^{-\frac{1}{D-3} \left[\sqrt{\frac{(p+1)(D-2)}{(N-2)}} + (D-4) \right]} \left[dy^2 + y^2 \left(1 - \left(\frac{y_0}{y} \right)^{D-3} \right) d\Omega_{D-2}^2 \right] \end{aligned} \quad (5.15)$$

Note that the extremal limit of this solution now corresponds to $y_0 = 0$

It is obviously interesting to contrast these solutions with those of Horowitz and Strominger [2]. Recall that the magnetically charged HS solutions have the form

$$e^{-2\phi} = \left[1 - \left(\frac{r_-}{r} \right)^{D-3} \right]^{\frac{-2}{\beta}} \quad (5.16)$$

$$\begin{aligned} d\tilde{s}^2 &= \left[1 - \left(\frac{r_+}{r} \right)^{D-3} \right] \left[1 - \left(\frac{r_-}{r} \right)^{D-3} \right]^{\gamma_x - 1} dt^2 - \left[1 - \left(\frac{r_-}{r} \right)^{D-3} \right]^{\gamma_x} dx_i^2 \\ &\quad - \left[1 - \left(\frac{r_+}{r} \right)^{D-3} \right]^{-1} \left[1 - \left(\frac{r_-}{r} \right)^{D-3} \right]^{\gamma_r} dr^2 - r^2 \left[1 - \left(\frac{r_-}{r} \right)^{D-3} \right]^{\gamma_r + 1} d\Omega_{D-2}^2 \end{aligned} \quad (5.17)$$

where the exponents are given by

$$\gamma_x = \frac{4\alpha + D - 3}{8\alpha\beta} \quad ; \quad \gamma_r = \frac{1}{2\alpha\beta} \left(\alpha + \frac{(D-11)}{4} \right) - \frac{(D-5)}{(D-3)} \quad (5.18)$$

and

$$Q^2 = \frac{(D-3)^2 (r_+ r_-)^{D-3}}{4\alpha\beta} \quad (5.19)$$

in terms of our constants α and β with $N = 10$.

Obviously these solutions are rather different, as is expected, since the HS solutions do not have boost symmetry, however, since the HS solutions *are* boost symmetric in their extremal limit, we would expect that they would agree with ours. Indeed, the dilaton

solutions, (5.11) and (5.16), do give the same extremal solution, with (5.12) and (5.19) agreeing upon the charge for that solution. To compare the metrics, we must conformally transform (5.17) via (4.2), remembering that $N = 10$, before comparing with (5.13), which does indeed give the same extremal limit.

In general, the extremal limit of our solution is

$$\begin{aligned}
ds_e^2 &= \left(1 - \left(\frac{r_+}{r}\right)^{D-3}\right)^{\frac{8(D-3)}{\alpha\beta(N-2)^2}} (dt^2 - dx_i^2) - \left(1 - \left(\frac{r_+}{r}\right)^{D-3}\right)^{\frac{-8(p+1)}{\alpha\beta(N-2)^2} - \frac{2(D-4)}{D-3}} dr^2 \\
&\quad - r^2 \left(1 - \left(\frac{r_+}{r}\right)^{D-3}\right)^{\frac{2}{(D-3)} - \frac{8(p+1)}{\alpha\beta(N-2)^2}} d\Omega_{D-2}^2 \\
&= \left(1 + \left(\frac{r_+}{y}\right)^{D-3}\right)^{\frac{-8(D-3)}{\alpha\beta(N-2)^2}} (dt^2 - dx_i^2) - \left(1 + \left(\frac{r_+}{y}\right)^{D-3}\right)^{\frac{8(p+1)}{\alpha\beta(N-2)^2}} [dy^2 + y^2 d\Omega_{D-2}^2]
\end{aligned} \tag{5.20}$$

These are the metrics for extremally magnetically charged p -branes in arbitrary dimension. Some of these metrics have already been derived by Duff and Lu[26], who found these solutions for specific values of α , however, note here that we have no restriction on the value of α .

We can therefore be reasonably confident that our solutions do represent true boost-symmetric p -branes. Clearly they lack the relative simplicity of the other GMHS solutions, they are rather messy and still suffer from singularities. Whether or not they can be convincingly shown to be the far field limit of some soliton or topological defect requires not only a choice of model, but also a decision on how to couple that matter to the string dilaton - a problem we will not address in this paper. However, we do believe that in principle, there is no obstruction to painting a p -brane core onto the spacetime to smooth out the singularity.

As before, we would like to consider some specific examples. Let $a = -1$, then $\alpha = \frac{-2(D-3)}{(N-2)}$, and it turns out that $\beta = -2$ for all N and D . We also prefer to invert the conformal transformation (4.2) to quote results in the possibly more familiar string frame. The dilaton solution is given by (5.11) with $\beta = -2$, and the metric in the string frame is

$$\begin{aligned}
d\tilde{s}^2 &= \left(\frac{r^{D-3} - r_+^{D-3}}{r^{D-3} - r_-^{D-3}}\right)^{\frac{l}{p+1}} (dt^2 - dx_i^2) - e^{\frac{4\phi}{(D-3)}} \left(\frac{r^{D-3} - r_+^{D-3}}{r^{D-3} - r_-^{D-3}}\right)^{\frac{-(l+D-4)}{D-3}} \times \\
&\quad \left[\left(1 - \left(\frac{r_-}{r}\right)^{D-3}\right)^{\frac{-2(D-4)}{(D-3)}} dr^2 + r^2 \left(\frac{r^{D-3} - r_+^{D-3}}{r^{D-3} - r_-^{D-3}}\right) \left(1 - \left(\frac{r_-}{r}\right)^{D-3}\right)^{\frac{2}{D-3}} d\Omega_{D-2}^2 \right]
\end{aligned} \tag{5.21}$$

For $D = 5$, $N = 10$ we get the 5-brane: $n = 5$, $m = 11/8$, $l = 3/2$, and

$$d\tilde{s}^2 = \left(\frac{r^2 - r_+^2}{r^2 - r_-^2} \right)^{1/4} (dt^2 - dx_i^2) - e^{2\phi} \left(\frac{r^2 - r_+^2}{r^2 - r_-^2} \right)^{-5/4} \left(1 - \frac{r_-^2}{r^2} \right)^{-1} \left[dr^2 + r^2 \left(1 - \frac{r_+^2}{r^2} \right) \left(1 - \frac{r_-^2}{r^2} \right) d\Omega_{III}^2 \right] \quad (5.22)$$

with dilaton

$$e^{2\phi} = 1 + \frac{2r_-^2}{3(r_+^2 - r_-^2)} \left[1 - \left(\frac{r^2 - r_+^2}{r^2 - r_-^2} \right)^{\frac{3}{2}} \right] \quad (5.23)$$

This is the metric and dilaton of the non-supersymmetric 5-brane.

For $D = 4$ and $N = 10$, we have the 6-brane of heterotic string gravity:

$$d\tilde{s}^2 = \left(\frac{r - r_+}{r - r_-} \right)^{\frac{1}{2\sqrt{7}}} (dt^2 - dx_i^2) - e^{4\phi} \left(\frac{r - r_+}{r - r_-} \right)^{\frac{-\sqrt{7}}{2}} [dr^2 + (r - r_+)(r - r_-)d\Omega_{II}^2] \quad (5.24)$$

with dilaton

$$e^{2\phi} = 1 + \frac{2r_-}{\sqrt{7}(r_+ - r_-)} \left[1 - \left(\frac{r - r_+}{r - r_-} \right)^{\frac{\sqrt{7}}{2}} \right] \quad (5.25)$$

6. Dual and Self-Dual Branes.

The solutions of the previous two sections were all magnetically charged, obviously we would like to find electrically charged solutions. Fortunately, a generalisation of the duality transformation of Horowitz and Strominger allows us to derive these solutions. Define

$$K = e^{2\alpha\phi} * F \quad (6.1)$$

where $*$ is the Hodge dual, then (4.6a) is equivalent to the statement that K is a closed $(N - D + 2)$ -form. We can then see that

$$K^2 = (-)^{N-1} e^{4\alpha\phi} F^2 \quad (6.2)$$

Then, provided we write $D' = N + 4 - D$ and $\alpha' = -\alpha$, the energy-momentum tensor (4.8), the dilaton equation (4.9), and hence the Einstein equations (4.10) are invariant under the operation $\{F, D, \alpha\} \rightarrow \{K, D', \alpha'\}$. Thus to get an electrically charged solution, we take

the magnetic solution for $N + 4 - D$ and $-\alpha$, and dualise the magnetic form according to (6.1).

For example, if we dualize the 2-form F_{ab} of heterotic string gravity, then we get a 0-brane, or black hole, in ten dimensions:

$$d\tilde{s}^2 = \left(1 - \left(\frac{r_+}{r}\right)^7\right) \left(1 - \left(\frac{r_-}{r}\right)^7\right) dt^2 - \frac{dr^2}{\left(1 - \left(\frac{r_+}{r}\right)^7\right) \left(1 - \left(\frac{r_-}{r}\right)^7\right)^{5/7}} - r^2 \left(1 - \left(\frac{r_-}{r}\right)^7\right)^{2/7} d\Omega_8^2 \quad (6.3)$$

with dilaton

$$e^{2\phi} = 1 - \left(\frac{r_-}{r}\right)^7 \quad (6.4)$$

which, not surprisingly, is identical to the solution in [2].

However, given the motivation of string theory in our choice of action (4.1) it is interesting to ask what a string or 1-brane solution in ten dimensions will look like, since it is believed that the extremal limit[27] of such an electrically charged solution actually *is* a fundamental string. An electrically charged 1-brane has $D' = 9$, or $D = 5$, and hence corresponds to the axion field, H_{abc} , carrying “electric” charge. We take $a = -1$ in the original action (4.1) which corresponds to $\alpha = -1/2$. Therefore, to get our family of strings, we dualize the magnetic $D = 9, \alpha = 1/2$ solutions. Choosing the y -coordinate as defined in (5.14), we may read off our solution from (5.15), (5.11) as:

$$ds^2 = \Lambda(dt^2 - dx_i^2) - \Lambda^{-1/3} \left[dy^2 \left(1 - \left(\frac{y_0}{y}\right)^6\right)^{-5/6} + y^2 \left(1 - \left(\frac{y_0}{y}\right)^6\right)^{1/6} d\Omega_7^2 \right] \quad (6.5)$$

where

$$\begin{aligned} \Lambda &= e^{3\phi/2} \left(1 - \left(\frac{y_0}{y}\right)^6\right)^{\sqrt{7}/4} \\ e^{-2\phi} &= 1 + \frac{2r_-^6}{\sqrt{7}y_0^6} \left[1 - \left(1 - \left(\frac{y_0}{y}\right)^6\right)^{\sqrt{7}/2} \right] \end{aligned} \quad (6.6)$$

and the axion field is given by:

$$H_{abc} = Qe^{3\phi} * \epsilon_7 \quad (6.7)$$

in the Einstein frame. To get to the string frame

$$\begin{aligned} \tilde{g}_{ab} &= e^{\phi/2} g_{ab} \\ \tilde{H}_{abc} &= e^{-\phi} H_{abc} \end{aligned} \quad (6.8)$$

which gives for the metric

$$ds^2 = e^{2\phi} \left(1 - \left(\frac{y_0}{y} \right)^6 \right)^{\sqrt{7}/4} (dt^2 - dx_i^2) - \left(1 - \left(\frac{y_0}{y} \right)^6 \right)^{-\sqrt{7}/12} \left[dy^2 \left(1 - \left(\frac{y_0}{y} \right)^6 \right)^{-5/6} + y^2 \left(1 - \left(\frac{y_0}{y} \right)^6 \right)^{1/6} d\Omega_7^2 \right] \quad (6.9)$$

It can be seen that the extremal limit is indeed the fundamental string of [27].

So far, we have been examining either electrically or magnetically charged branes, but if the spacetime dimension, N , is even, then it is possible that the charge be a self-dual (or anti-self-dual) $N/2$ -form. For such sources,

$$F^2 = (-)^{N/2} F^2 \quad (6.10)$$

and therefore if $N = 4j + 2$ for some $j \in \mathbb{Z}$, then F^2 vanishes, and the dilaton decouples from the equations of motion. Taking F to be the self dual form

$$F = \frac{Q}{\sqrt{2}} [\epsilon_{N/2} + * \epsilon_{N/2}] \quad (6.11)$$

gives the energy-momentum tensor (4.8) with ϕ set to zero. Hence the self-dual $\frac{N}{2} - 2$ -branes are given by the $\alpha = 0$ solutions of section five. (In reality, $\alpha = a$.)

$$ds^2 = \Xi(dt^2 - dx_i^2) - \Xi^{-1} \left(1 - \left(\frac{r_+}{r} \right)^{\frac{N-2}{2}} \right)^{\frac{2}{N-2}} \left(1 - \left(\frac{r_-}{r} \right)^{\frac{N-2}{2}} \right)^{\frac{2}{N-2}} \left[\frac{dr^2}{\left(1 - \left(\frac{r_+}{r} \right)^{\frac{N-2}{2}} \right) \left(1 - \left(\frac{r_-}{r} \right)^{\frac{N-2}{2}} \right)} + r^2 d\Omega_{N/2}^2 \right] \quad (6.12)$$

where

$$\Xi = \left(\frac{r_+^{\frac{N-2}{2}} - r_+^{\frac{N-2}{2}}}{r_+^{\frac{N-2}{2}} - r_-^{\frac{N-2}{2}}} \right)^{\frac{\sqrt{N}}{N-2}} \left[1 + \frac{2r_-^{\frac{N-2}{2}}}{\sqrt{N}(r_+^{\frac{N-2}{2}} - r_-^{\frac{N-2}{2}})} \left(1 - \left(\frac{r_+^{\frac{N-2}{2}} - r_+^{\frac{N-2}{2}}}{r_+^{\frac{N-2}{2}} - r_-^{\frac{N-2}{2}}} \right)^{\frac{\sqrt{N}}{2}} \right) \right]^{\frac{-4}{N-2}} \quad (6.13)$$

7. Discussion.

We have derived the metrics for a boost symmetric p -branes in an arbitrary number of spacetime dimensions. We use an Einstein-Maxwell-Dilaton action for arbitrary values

of the dilaton coupling and arbitrary values of the charge carried by the ‘Maxwell’ field. Previously, the only boost symmetric metrics considered were those with maximal, or extremal, charge. We find that the exact solutions generically have naked singularities at the brane core, however, we believe that this is an aspect of the idealised solution and that is one were to consider instead the branes as arising as defects in some spontaneously broken gauge theory, this unpleasant behaviour would be smoothed out by the internal structure of the defect. We have shown in principle that there is no obstruction to doing this.

An alternative to placing a source at the p -brane core is instead to regard these solutions as leading order solutions to string gravity, and therefore the singularity might get smoothed out by higher order stringy effects (see e.g. [28,29]). In other words, string theory might act as a cosmic censor. Since the bulk of the solutions presented here are not extremal, and hence not supersymmetric or BPS states, they would certainly be expected to acquire higher order or loop corrections in a fully consistent expansion. However, we should point out that while the solutions presented here display the same strong-weak/electric-magnetic duality present in previous results, the main difference is that the singularity actually sits at r_+ , rather than lying inside some event horizon. The choice of boundary conditions for the dilaton means that the dilaton is always finite, whether the brane is magnetically or electrically charged, at this point - except in the extremal limit. In other words, if we identify e^ϕ as our string loop coupling, then the singularities always occur in a weakly, or at least not strongly, coupled régime. Therefore we can only appeal to $O(\alpha')$ arguments[28]. Whether or not such a process would smooth out the singularity is an open question, indeed, whether one should smooth out singularities in general is an open question[30]! However, having a naked singularity is rather disturbing for the more deterministic!

One interesting facet of the HS solutions is that they are unstable[3] except possibly in their extremal limit. This is interesting precisely because it is only in the extremal limit that these solutions are boost symmetric. It would be useful to know if the solutions presented here are stable. Indeed, since we are arguing that they correspond to the far field of some topological defect, it would be very surprising if they did exhibit a linear instability. If they are stable, then they could resolve the question of the endpoint of the instability in [3]. One of the curiosities of the instability was that it appeared to lead to fragmentation of the event horizon, this in turn would lead to a violation of cosmic censorship. While there was no obstruction, other than the mythical cosmic censor, to this occurring for the uncharged solutions, it did seem that the magnetically charged solutions should not be able to break, for the simple reason that the charge is topological, and there

would be nowhere for it to go. It has been suggested that there was some endpoint other than fragmentation for this instability[†]. Perhaps the p -branes presented here represent that endpoint?

Of course, as we remarked in the introduction, topology does not guarantee stability, for there might be a higher dimensional generalisation of the C-metric which would provide an instanton for the decay of some or all of the p -branes. Briefly, in four-dimensions, the C-metric corresponds to two black holes accelerating apart, attached to infinity by conical deficits responsible for their acceleration. It was shown in [10] that these deficits could be smoothly replaced by a Nielsen-Olesen vortex, and hence that an otherwise stable topological defect could decay as suggested in [7-9]. Suppose one generalises the C-metrics to higher dimensions, then one has a process for the decay of an idealized p -brane in N -dimensions. Given such an instanton, it seems likely that at least some of the solutions found here would be susceptible to just such a non-perturbative decay process. It is notable that the decay of strings in four-dimensions is insensitive to whether the vortex is sitting at its Bogomolnyi point or not. Therefore, if there is a non-perturbative decay process which is indifferent to BPS states, it is possible that some of these states might also be unstable. Clearly, if certain extremal states are considered to be of importance in stringy duality arguments, testing their resilience to non-perturbative decay processes is crucial. We have provided a first step in that direction here by generalizing the conical singularity to higher dimensions. It would be amusing, although admittedly unlikely, if the extremal black strings were unstable to non-perturbative topology changing processes. Often, supersymmetry is invoked for protection against instability, for example, in the decay of the Kaluza-Klein vacuum[31]. However, in an intriguing recent paper concerning the decay of Kaluza-Klein magnetic universes[32], it was shown that while supersymmetry did protect against a decay of the Witten type[31], it did not protect against decay of the black hole pair nucleation type. Such a result does not bode well for the decay of strings in higher dimensions.

Acknowledgements.

I would like to thank Peter Bowcock and Jeff Harvey for useful comments and suggestions. This work was supported by the Royal Society.

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[†] I would like to thank G.Gibbons and G.Horowitz for discussions on this point.

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